JOULE-THOMSON AND JOULE EFFECTS FOR BOSE-EINSTEIN AND FERMI-DIRAC GAS*

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ABSTRACT. The paper gives results for the Joule-Thomson effect and adiabatic process for a degenerate Bose-Einstein gas. A general formula for non-degenerate matter (both Fermi-Dirac and Bose-Einstein) is given and the region of transition from degeneracy to non-degeneracy in the Bose-Einstein gas is discussed in detail.

The second part of the paper discusses the Joule effect for the various cases of matter obeying either statistics.

Various properties of degenerate matter (degenerate in the sense of Fermi-Dirac and Bose-Einstein statistics) have been investigated in the last decade and a half, and the results have found numerous applications in various physical and astrophysical problems. Recently Kothari [1] discussed the Joule-Thomson effect and adiabatic process for a non-relativistic Fermi-Dirac degenerate gas, and Srivastava [2] has considered the non-relativistic non-degenerate cases of Fermi-Dirac statistics. Gogate [3] has extended the discussion to relativistic Fermi-Dirac degenerate and non-degenerate gas. However, the dicussion of Joule-Thomson effect and adiabatic process is incomplete, because the degenerate Bose-Einstein case has not been treated so far. London's theory of liquid He II has revived the interest in the properties of Bose-Einstein degenerate gas. The first part of the present paper deals with the Joule-Thomson effect and adiabatic change in Bose-Einstein gas. A general result for non-degeneracy has also been derived. In the second part the Joule effect has been calculated for both the statistics. It has been shown for both the phenomenon of Joule-Thomson effect and Joule effect that a Fermi-Dirac gas always gets heated and a Bose-Einstein gas is always cooled. For matter obeying Bose-Einstein statistics the region of transition from degeneracy to non-degeneracy has been studied in detail.

PART I—JOULE-THOMSON EFFECT

The non-degenerate case as mentioned already has been treated by Srivastava but here we shall derive a series expansion for the change in temperature produced by the throttling process.

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Classical thermodynamics gives the well-known relation

$$\left(\frac{\partial T}{\partial p}\right)_H = \frac{T\left(\frac{\partial V}{\partial T}\right)_p - V}{C_p} \qquad \dots \quad (1)$$

Let us consider an assembly of N particles occupying a volume V. Taking into account relativity mechanics the number of particles possessing the kinetic energy ϵ and lying in the range ϵ to $\epsilon+d\epsilon$ is given by

$$N(\epsilon)d\epsilon = \frac{a(\epsilon)}{\frac{1}{A}e^{\epsilon/kT}+\beta}, \qquad \dots \quad (2)$$

where $$a(\epsilon) = \frac{4\pi gV}{c^3h^3}(\epsilon^2+2mc^2\epsilon)^{\frac{1}{2}}(\epsilon+mc^2)d\epsilon.$$

Hence E the total kinetic energy for the assembly is given by

$$E = \int_0^\infty \frac{4\pi gV}{c^3h^3} \frac{(\epsilon^2+2mc^2\epsilon)^{\frac{1}{2}}(\epsilon+mc^2)\epsilon}{\frac{1}{A}e^{\epsilon/kT}+\beta} d\epsilon. \qquad \dots \quad (3)$$

However, considerable simplification is achieved by considering two extreme cases: (*i*) *completely non-relativistic case* and (*ii*) *completely relativistic case*. Equation (3) then reduces to

$$E = CV(kT)^{s+1}\int_0^\infty \frac{u^s}{\frac{1}{A}e^u+\beta}du, \qquad \dots \quad (4)^*$$

and also, if N denotes the total number of particles in the assembly (we shall take N to be equal to the Avogadro number, *i.e.*, $R=kN$), we have

$$N = CV(kT)^s\int_0^\infty \frac{u^{s-1}du}{\frac{1}{A}e^u+\beta} \qquad \dots \quad (4a)^*$$

where $u=\frac{\epsilon}{kT}$. In the non-relativistic case

$$s=3/2 \quad \text{and} \quad C=\frac{2\pi g(2m)^{\frac{3}{2}}}{h^3}, \qquad \dots \quad (5)$$

and in the relativistic case

$$s=3 \quad \text{and} \quad C=\frac{4\pi g}{c^3h^3}. \qquad \dots \quad (6)$$

* These follow immediately by assuming a general distribution law of the form

$$N(\epsilon)d\epsilon = CV(kT)^s\frac{u^{s-1}}{\frac{1}{A}e^u+\beta}du.$$

TABLE I

		Fermi-Dirac Statistics	Bose-Einstein Statistics
Non-degeneracy	(1) Non-relativistic	$E_+=\frac{3}{2}RT\left[1+\frac{A_0}{2^{\frac{5}{2}}}-A_0^2\left(\frac{2}{3^{\frac{5}{2}}}-\frac{1}{4^{\frac{3}{2}}}\right)+A_0^3\left(\frac{3}{4^{\frac{5}{2}}}+\frac{5}{2^{\frac{11}{2}}}-\frac{3}{6^{\frac{3}{2}}}\right)\ldots\right]$	$E_-=\frac{3}{2}RT\left[1-\frac{A_0}{2^{\frac{5}{2}}}-A_0^2\left(\frac{2}{3^{\frac{5}{2}}}-\frac{1}{4^{\frac{3}{2}}}\right)-A_0^3\left(\frac{3}{4^{\frac{5}{2}}}+\frac{5}{2^{\frac{11}{2}}}-\frac{3}{6^{\frac{3}{2}}}\right)\ldots\ldots\right]$
	(2) Relativistic	$E_+=3RT\left[1+\frac{A_0^R}{2^4}-A_0^{R^2}\left(\frac{2}{3^4}-\frac{1}{4^3}\right)+A_0^{R^3}\left(\frac{3}{4^4}+\frac{5}{4^5}-\frac{3}{6^3}\right)\ldots\right]$	$E_-=3RT\left[1-\frac{A_0^R}{2^4}-A_0^{R^2}\left(\frac{2}{3^4}-\frac{1}{4^3}\right)-A_0^{R^3}\left(\frac{3}{4^4}+\frac{5}{4^5}-\frac{3}{6^3}\right)\ldots\ldots\right]$
Degeneracy	(1) Non-relativistic	$E_-=\frac{2\pi gVh^2}{5m}\left(\frac{3n}{4\pi g}\right)^{\frac{5}{3}}\left[1+\frac{5\,2^{\frac{2}{3}}\pi^{\frac{4}{3}}}{3^{\frac{7}{3}}A_0^{\frac{4}{3}}}+\ldots\ldots\right]$	$E_-=\Gamma(\tfrac{5}{2})CV(kT)^{\frac{5}{2}}\zeta(\tfrac{5}{2})$
	(2) Relativistic	$E_-=\pi gchV\left(\frac{3n}{4\pi g}\right)^{\frac{4}{3}}\left[1+\frac{\pi^2 2^{\frac{1}{3}}}{3^{\frac{5}{3}}(A_0^R)^{\frac{2}{3}}}+\ldots\ldots\right]$	$E_-=\Gamma(4)\,CV(kT)^4\zeta(4).$

where, $A_0=\frac{nh^3}{g(2\pi mkT)^{\frac{3}{2}}}$, $A_0^R=\frac{n}{8\pi g}\left(\frac{ch}{kT}\right)^3$, and $\zeta(k)=1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}\ldots\ldots$

The results of integration of equation (4) for the different cases of degeneracy and non-degeneracy are shown in Table 1.

In the non-degenerate case, *i.e.*, $\Lambda << 1$, and we have, by integrating equation (4)* and (4*a*),

$$E_{+} = sNkT[1 + a\beta A_1 - b\beta^2 A_1^2 + c\beta^3 A_1^3 \ldots\ldots\ldots], \qquad \ldots \quad (7)$$

where $\quad a = \dfrac{1}{2^{s+1}}, \quad b = \left(\dfrac{2}{3^{s+1}} - \dfrac{1}{4^s}\right), \quad c = \dfrac{3}{4^{s+1}} + \dfrac{5}{2^{3s+1}} - \dfrac{3}{6^s},$

and

$$A_1 \equiv \frac{N}{\Gamma s C V (kT)^s} = \sum_{n=1}^{n=\infty} \frac{A^n}{n^s} (-\beta)^{n-1}. \qquad \ldots \quad (8)\dagger$$

Equation (7) gives the general expression for energy in non-degeneracy and by giving appropriate values to β and s we obtain the different cases, *viz.*,

(*i*) $\beta = +1, \ s = \frac{3}{2}$ *completely non-relativistic* Fermi-Dirac non-degenerate case.

(*ii*) $\beta = +1, \ s = 3$ *completely relativistic* Fermi-Dirac non-degenerate case.

(*iii*) $\beta = -1, \ s = \frac{3}{2}$ *completely non-relativistic* Bose-Einstein non-degenerate case.

(*iv*) $\beta = -1, \ s = 3$ *completely relativistic* Bose-Einstein non-degenerate case.

We have, since‡ $E = spV, \qquad \ldots \quad (9)$

$$p_{+} = \frac{RT}{V}[1 + a\beta A_1 - b\beta^2 A_1^2 + c\beta^3 A_1^3 + \ldots\ldots\ldots] \qquad .. \quad (10)$$

and, as from (8) we have

$$\left(\frac{\partial A_1}{\partial T}\right)_v = -\frac{s}{T} A_1,$$

we obtain $\quad C_{v+} = \left(\dfrac{\partial E_{+}}{\partial T}\right)_v$

$$= Rs[1 - (s-1)a\beta A_1 + (2s-1)b\beta^2 A_1^2 - (3s-1)c\beta^3 A_1^3 \ldots\ldots] \ldots \quad (11)$$

where $\quad R = Nk$.

* The suffix + attached to a quantity denotes its value in the non-degenerate case. The suffix − represents the degenerate case.

† A_1 becomes equal to A_0 of the previous papers in the non-relativistic case, and in the relativistic case to A_0^R.

‡ For an ideal gas we have the usual relations, $E = \frac{3}{2}pV$ in the non-relativistic case and $E = 3pV$ in the relativistic case.

Again, since

$$\left(\frac{\partial V}{\partial T}\right)_p = -\frac{\left(\frac{\partial p}{\partial T}\right)_v}{\left(\frac{\partial p}{\partial V}\right)_T}, \qquad \dots \quad (12)$$

we have substituting from (10)

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{V}{T}\left[\frac{1-(s-1)a\beta A_1+(2s-1)b\beta^2 A_1^2-(3s-1)c\beta^3 A_1^3 \dots\dots\dots}{1+2a\beta A_1-3b\beta^2 A_1^2+4c\beta^3 A_1^3 \dots\dots\dots\dots\dots\dots}\right],$$

or
$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{V}{T}\left[1-(s+1)a\beta A_1+(2s+2)(a^2+b)\beta^2 A_1^2 - (s+1)(4a^3+7ab+3c)\beta^3 A_1^3+\dots\dots\dots\dots\dots\dots\right]. \qquad \dots \quad (13)$$

But we have,
$$C_p - C_v = T\left(\frac{\partial p}{\partial T}\right)_v\left(\frac{\partial V}{\partial T}\right)_p, \qquad \dots \quad (14)$$

and hence using (9) and (13), we have

$$C_p = C_v\left\{1+\frac{T}{sV}\left(\frac{\partial V}{\partial T}\right)_p\right\}, \qquad \dots \quad (14a)$$

or
$$C_{p_1} = C_{v_1}\frac{(s+1)}{s}\{1-a\beta A_1+2(a^2+b)\beta^2 A_1^2-(4a^3+7ab+3c)\beta^3 A_1^3 \dots\} \qquad (15)$$

And therefore from equations (1), (11), (13) and (15) we obtain

$$\left(\frac{\partial T}{\partial p}\right)_H = -\frac{V}{R}\left[\frac{a\beta A_1-2(a^2+b)\beta^2 A_1^2+(4a^3+7ab+3c)\beta^3 A_1^3 \dots\dots\dots\dots}{\{1-a\beta A_1+2(a^2+b)\beta^2 A_1^2+\dots\}\{1-(s-1)a\beta A_1+(2s-1)\,b\beta^2 A_1^2\dots\}}\right]$$

$$= -\frac{V}{R}\left[a\beta A_1+\{(s-2)a^2-2b)\}\beta^2 A_1^2+\{(s^2-3s+3)a^2-2(2s-3).+3c\}\beta^3 A_1^3 \dots\dots\dots\dots\right], \qquad (16)$$

This is the general result for the non-degenerate case and, to a first order, this gives the same results as obtained by Srivastava and Gogate (*loc. cit.*).

We now confine ourselves to the Bose-Einstein statistics only ($\beta = -1$). The non-degenerate case of Bose-Einstein statistics extends right up to $A_1 = \zeta(s)-0$, for the degenerate case A_1 being equal to or greater than $\zeta(s)$. However, for $A_1 = \zeta(s)-0$ the series in equation (16) is not sufficiently convergent to give accurate results. We have, therefore, to use a direct method to determine the limiting value of $\left(\frac{\partial T}{\partial p}\right)_H$ and its rate of change with respect to A_1 at $A_1 = \zeta(s)-0$. This is done as follows.

From equation (4), we have, since $\beta = -1$

$$E_{+} = \frac{RsT}{A_1}\sum_{n=1}^{n=\infty}\frac{A^n}{n^{s+1}}, \qquad \ldots \quad (17)$$

where
$$A_1 = \frac{N}{\Gamma sCV(kT)^s} = \sum_{n=1}^{n=\infty}\frac{A^n}{n^s}, \qquad \ldots \quad (18)$$

and therefore,
$$\frac{T}{sV}\left(\frac{\partial A}{\partial T}\right)_v = \left(\frac{\partial A}{\partial V}\right)_T. \qquad \ldots \quad (18a)$$

This gives
$$\left(\frac{\partial A_1}{\partial T}\right)_v = \frac{-sA_1}{T}, \quad \left(\frac{\partial^2 A_1}{\partial T^2}\right)_v = \frac{s(s+1)A_1}{T^2} \qquad \ldots \quad (19)$$

and
$$\left(\frac{\partial A_1}{\partial T}\right)_v = \Sigma\frac{A^{n-1}}{n^{s-1}}\left(\frac{\partial A}{\partial T}\right)_v, \qquad \ldots \quad (20)$$

also from (18),
$$\left(\frac{dA}{dA_1}\right)_1 = \frac{1}{\zeta(s-1)}, \qquad \ldots \quad (20a)$$

where the *suffix 1 attached to a function denotes its value* at* A $=\zeta(s)-0$. If T_0 is the value of T at this point, then we have

$$\left(\frac{\partial A_1}{\partial T}\right)_{1v} = \frac{-s\zeta(s)}{T_0}, \qquad \ldots \quad (21)$$

and from (19) and (20*a*) we obtain

$$\left(\frac{\partial A}{\partial T}\right)_{1v} = -\frac{s}{T_0}\frac{\zeta(s)}{\zeta(s-1)}. \qquad \ldots \quad (22)$$

We can now derive an expression for Cv_1. From equation (17) we have

$$C_{v+} = \left(\frac{\partial E_+}{\partial T}\right)_v = R\left\{\frac{(s+1)s}{A_1}\Sigma\frac{A^n}{n^{s+1}} + \frac{sT}{A_1}\left(\frac{\partial A}{\partial T}\right)_v\Sigma\frac{A^{n-1}}{n^s}\right\}, \qquad \ldots \quad (23)$$

or, from (20),
$$C_{v+} = Rs\left\{\frac{(s+1)}{A_1}\Sigma\frac{A^n}{n^{s+1}} - s\Sigma\frac{A^{n-1}}{n^s}\Big/\Sigma\frac{A^{n-1}}{n^{s-1}}\right\}, \qquad \ldots \quad (24)$$

and, therefore,
$$Cv_1 = Rs\left\{(s+1)\frac{\zeta(s+1)}{\zeta(s)} - \frac{s\zeta(s)}{\zeta(s-1)}\right\}. \qquad \ldots \quad (25)$$

We shall now determine Cp_1. The relation between C_p and C_v is given by (14*a*).

Since A is a function of T and V, we have

$$\left\{\frac{\partial A}{\partial T}\right\}_p = \left\{\frac{\partial A}{T}\right\}_v + \left\{\frac{\partial A}{\partial V}\right\}_T\left\{\frac{\partial V}{\partial T}\right\}_p. \qquad \ldots \quad (26)$$

* There should not be any confusion as regards A_1 which has already been defined by equation (8).

The values of $\left\{\frac{\partial A}{\partial V}\right\}_{1T}$ and $\left\{\frac{\partial A}{\partial T}\right\}_{1v}$ are obtained from equations (18*a*) and (22).

In order to obtain $\left\{\frac{\partial A}{\partial T}\right\}_{p}$, we make use of the equation

$$p_1 = \frac{C(kT)^s}{s}\int \frac{\epsilon^s d\epsilon}{\frac{e^{\epsilon/kT}}{A}-1} \qquad \dots \quad (27)$$

which follows from (4) and (9). Differentiating this with respect to T and keeping p constant, we have

$$\frac{(s+1)}{T}\Gamma s+1\,\Sigma\frac{A^n}{n^{s+1}} + \frac{s}{A}\left\{\frac{\partial A}{\partial T}\right\}_p \Gamma s\,\Sigma\frac{A^n}{n^s} = 0. \qquad \dots \quad (28)$$

When $A_1=\zeta(s)-0$, we have, $\left\{\frac{\partial A}{\partial T}\right\}_{1p} = \frac{-(s+1)}{T}\,\frac{\zeta(s+1)}{\zeta(s)}$. $\qquad \dots \quad (29)$

Substituting the values of $\left\{\frac{\partial A}{\partial V}\right\}_{1T}$, $\left\{\frac{\partial A}{\partial T}\right\}_{1v}$ and $\left[\frac{\partial A}{\partial T}\right\}_{1p}$ in (26),

we obtain*

$$\left\{\frac{\partial V}{\partial T}\right\}_{1p} = \frac{V}{T_0}\left\{\frac{(s+1)\,\zeta(s+1)\,\zeta(s-1)}{[\zeta(s)]^2} - s\right\}, \qquad \dots \quad (30)$$

and hence, from (14*a*), $C_{p_1} = \frac{s+1}{s}\,\frac{\zeta(s+1)\,\zeta(s-1)}{[\zeta(s)]^2}C_{v_1}$. $\qquad \dots \quad (31)$

Therefore, substituting the above result in equation (1) and using (25), we finally obtain

$$\left\{\frac{\partial T}{\partial P}\right\}_{1H} = \frac{V}{R}\,\frac{\zeta(s)}{(s+1)\,\zeta(s+1)}\left[1-\frac{1}{\frac{(s+1)\,\zeta(s+1)\,\zeta(s-1)}{[\zeta(s)]^2}-s}\right], \qquad \dots \quad (32)$$

$$= \left\{\begin{matrix} 0{\cdot}7844\,\frac{V}{R} \text{ for } s=3/2, \\ 0{\cdot}1338\,\frac{V}{R} \text{ for } s=3. \end{matrix}\right\} \qquad \dots \quad (32a)$$

The rate of change of $\left\{\frac{\partial T}{\partial p}\right\}_H$ with respect to A_1 is determined in a similar way.

* $\left(\frac{\partial V}{\partial T}\right)_p$ could also be obtained by differentiating p obtained from equations (9) and (4) and using (12).

From equations (26), (28), (20) and (18*a*), we have

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{V}{T}\frac{(s+1)\sum\frac{A^n}{n^{s+1}}\sum\frac{A^n}{n^{s-1}}}{A_1^2} - \frac{Vs}{T}, \qquad (33)$$

and substituting this in equation (14*a*), we have

$$C_{p_1} = C_{v_1}\left[\frac{s+1}{s}\,\frac{\sum\frac{A^n}{n^{s+1}}\sum\frac{A^n}{n^{s-1}}}{A_1^2}\right]. \qquad \dots (34)$$

Therefore equations (1), (33) and (34) give

$$\left(\frac{\partial T}{\partial p}\right)_u = \frac{V}{R}\,\frac{1 - A_1^2\Big/\sum\frac{A^n}{n^{s+1}}\sum\frac{A^n}{n^{s-1}}}{\frac{(s+1)}{A_1}\sum\frac{A^n}{n^{s+1}} - sA_1\Big/\sum\frac{A^n}{n^{s-1}}}. \qquad \dots (35)$$

The above relation after differentiation with respect to A_1 and simplification with the help of equation (20*a*), leads to

$$\left[\frac{d\left\{\left(\frac{\partial T}{\partial p}\right)_u R/V\right\}}{d\ A_1}\right]_1 = \Bigg[\left\{(s+1)\frac{\zeta(s+1)}{\zeta(s)} - \frac{s\zeta(s)}{\zeta(s-1)}\right\}\left\{\frac{[\zeta(s)]^2\,\zeta(s-2)}{\zeta(s+1)\,[\zeta(s-1)]^3}\right.$$

$$+\frac{[\zeta(s)]^3}{[\zeta(s+1)\,\zeta(s-1)]^2} - \left.\frac{2\zeta(s)}{\zeta(s+1)\,\zeta(s-1)}\right\} - \left\{1 - \frac{[\zeta(s)]^2}{\zeta(s+1)\,\zeta(s-1)}\right\}\left\{\frac{s\,\zeta(s)\,\zeta(s-2)}{[\zeta(s-1)]^3}\right.$$

$$\left. - \frac{(s+1)\zeta(s+1)}{[\zeta(s)]^2} + \frac{1}{\zeta(s-1)}\right\}\Bigg]\left[(s+1)\frac{\zeta(s+1)}{\zeta(s)} - \frac{s\zeta(s)}{\zeta(s-1)}\right]^{-2}$$

$$= \left\{\begin{matrix} 0\cdot 5503 & \text{for} & s = 3/2, \\ \infty & \text{for} & s = 3. \end{matrix}\right\}. \qquad \dots (36)^*$$

We shall now take up the degenerate case of the Bose-Einstein statistics. This is characterised by $A = 1$ and $A_1 = \zeta(s)$. Attaching the suffix $-$ to denote the value of the functions in the degenerate case, we have for the total energy E_-, by integrating equation (4),

$$E_- = \Gamma\overline{s+1}\,CV(kT)^{s+1}\zeta(s+1) \qquad \dots (37)$$

and from equation (9)

$$p_- = \Gamma s C(kT)^{s+1}\zeta(s+1) \qquad \dots (38)$$

and, therefore, from (37),

$$C_{v_-} = \Gamma\overline{s+2}\,C(kT)^s kV\zeta(s+1). \qquad \dots (39)$$

* Noting that $\frac{\zeta(s-2)}{[\zeta(s-1)]^3} = \frac{1}{2\pi}$ in the case of $s = 3/2$.

It is clear from equation (38) that the pressure is independent of the volume, and

$$\left(\frac{\partial V}{\partial T}\right)_p = \infty.$$

Therefore,

$$\left(\frac{\partial T}{\partial p}\right)_H = \frac{1 - \dfrac{V}{T\left(\dfrac{\partial V}{\partial T}\right)_p}}{\dfrac{C_v}{T\left(\dfrac{\partial V}{\partial T}\right)_p} + \left(\dfrac{\partial p}{\partial T}\right)_v} = \frac{1}{\left(\dfrac{\partial p}{\partial T}\right)_v},$$

$$= \frac{T}{(s+1)p} = \frac{V}{R}\,\frac{A_1}{(s+1)\,\zeta(s+1)}, \qquad \dots \quad (40)$$

$$= \begin{cases} 2/5\,\dfrac{T}{p} & \text{for } s = 3/2 \text{ (non-relativistic case),*} \\ 1/4\,\dfrac{T}{p} & \text{for } s = 3 \text{ (relativistic case).} \end{cases}$$

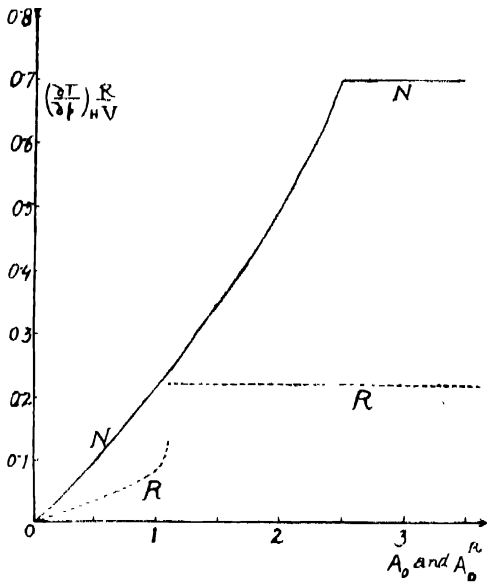

FIGURE 1

Joule-Thomson effect for a Bose-Einstein gas. Curve NN is for the non-relativistic case and RR for the relativistic case.

* These results, however, do not signify that the Joule-Thomson expansion can be used to produce cooling in a Bose-Einstein degenerate gas. For the amount of cooling is just equal to the difference in the temperature on the two sides of the throttle valve which are under different pressures, and the gas undergoing the throttling process simply acquires the temperature determined by the pressure on the side on which it emerges.

Equations (16) and (40) now enable us to obtain the values of $\left(\frac{\partial T}{\partial p}\right)_H$ R/V for the Bose-Einstein gas whether non-degenerate or degenerate. The curve NN in figure 1 shows the variation of $\left(\frac{\partial T}{\partial p}\right)_H$ R/V against A_0 for the non-relativistic case (A_0 denotes the value of A_1 in the non-relativistic case, $s=3/2$; and A_0'' denotes the value of A_1 in the relativistic case, $s=3$). The curve RR refers to the relativistic case, the abscissa for this curve being A_0''. The values of $\left(\frac{\partial T}{\partial p}\right)_H$ R/V at the point $A_1=\zeta(s)-0$ have been obtained from equation (32a) and its slope at $A_1=\zeta(s)-0$ from equation (36). In the non-relativistic case the angle made by the tangent to the curve at $A_0=\zeta(3/2)$ is approximately 28° 50′ and in the relativistic case the angle at $A_0''=\zeta(3)$ is 90°.* It will also be noticed, as is easily shown from equations (40) and (32a), that there is no discontinuity in the curve NN at $A_0=\zeta(3/2)$, but there is a discontinuity in the curve RR at $A_0''=\zeta(3)$.

Adiabatic change for a Bose-Einstein degenerate gas :—

In an adiabatic change the entropy remains constant. The entropy S is given by

$$S_{-}=(s+1)\Gamma s C V k (kT)^s \zeta(s+1). \qquad \dots \quad (41)$$

Consequently in an adiabatic change,

$$VT^s \text{ is constant,}$$

i.e.,

$$pV^{\frac{s+1}{s}} \text{ is constant (since } p \propto T^{s+1}). \qquad \dots \quad (41a)$$

PART II—JOULE EFFECT

2. When a gas expands without doing any external work, as in Joule's Experiment, there is no change in the internal energy of the system, *i.e.*,

$$dE = 0.$$

Using the well-known thermodynamic relations

$$dQ=dE+pdV$$
$$=C_v dT+l dV,$$

where

$$l=T\left(\frac{dp}{T}\right)_v,$$

* As in the figure the scales of the ordinate and abscissa are in the ratio of 5 : 1, the angles have been drawn equal to 70° and 90° respectively.

† See note on page 462 for definition of A_0.

we obtain, for the Joule effect,

$$\left(\frac{\partial T}{\partial V}\right)_E = \frac{p - T\left(\frac{\partial p}{\partial T}\right)_v}{C_v},$$

and using (9), we have,

$$\left(\frac{\partial T}{\partial V}\right)_E = \frac{p}{C_v} - \frac{T}{sV}. \qquad \dots \quad (42)$$

We shall now consider the non-degenerate case of Fermi-Dirac and Bose-Einstein statistics.

In this case p_+ and C_{v+} are given by equations (10) and (11) and substituting them in (42), we have

$$\left(\frac{\partial T}{\partial V}\right)_E = \frac{T}{V}\left[a\beta A_1 - \{2b-(s-1)a^2\}\beta^2 A_1^2 + \{3c - ab(4s-3) + (s-1)^2 a^3\}\beta^3 A_1^3 \dots\dots\right]. \qquad \dots \quad (43)$$

Considering to a first order only, we obtain, for a non-degenerate gas

(1) in the non-relativistic case

$$\left(\frac{\partial T}{\partial V}\right)_E = \frac{\beta T}{V} \cdot \frac{A_0}{2^{\frac{5}{2}}}, \qquad \dots \quad (44)$$

and (2) in the relativistic case

$$\left(\frac{\partial T}{\partial V}\right)_E = \frac{\beta T}{V} \cdot \frac{A_0^R}{16}, \qquad \dots \quad (44a)$$

where $\beta = 1$ for matter obeying Fermi-Dirac statistics and $\beta = -1$ for matter obeying Bose-Einstein statistics. It is clear from the above, since dV is always positive, that there is a heating effect in the Fermi-Dirac gas whereas a cooling is produced in the Bose-Einstein gas. Also the change is greater, the greater the degree of degeneracy.

(*b*) Now we shall consider degenerate matter. The degenerate case in the two statistics has to be treated separately. First we shall consider the Fermi-Dirac statistics. This is characterised by $A \gg 1$ and $\beta = 1$. The integrations are carried out by using Sommerfelds' formula according to which for large A,

$$\int_0^\infty \frac{u^s}{\frac{e^u}{A} + 1}\, du = \frac{u_0^{s+1}}{s+1}\left[1 + \frac{\pi^2}{6u_0^2} s(s+1) + \dots\right],$$

where $u_0 = \log A$.

Integrating equation (4) for large A, we have

$$E_- = \frac{s}{s+1} NkT(\Gamma s + 1 A_1)^{\frac{1}{s}}\left[1 + \frac{\pi^2}{6} \frac{(s+1)}{(\Gamma s + 1 A_1)^{\frac{2}{s}}} \dots\right],$$

which gives, since $E = spV$,

$$p_- = \frac{NkT}{(s+1)V} (\Gamma\overline{s+1}A_1)^{\frac{1}{s}} \left[1 + \frac{\pi^2}{6} \frac{(s+1)}{(\Gamma\overline{s+1}A_1)^{\frac{2}{s}}} \cdots \right],$$

and

$$C_{v-} = \left(\frac{\partial E_-}{\partial T} \right)_v = \frac{s\pi^2}{3} \frac{Nk}{(\Gamma\overline{s+1}A_1)^{\frac{1}{s}}} .$$

Substituting these values in equation (42), we have

$$\left(\frac{\partial T}{\partial V} \right)_{E_-} = \frac{T}{sV} \left\{ \frac{3(\Gamma\overline{s+1}A_1)^{\frac{2}{s}}}{(s+1)\pi^2} \cdots - 1 \right\}. \qquad \cdots \quad (45)$$

This is the general result for matter degenerate in the sense of Fermi-Dirac statistics. If we have put $s = 3/2$ we obtain to a first order, for the non-relativistic case,

$$\left(\frac{\partial T}{\partial V} \right)_{E_-} = \frac{T}{V} \frac{3^{\frac{4}{3}} A_0^{\frac{4}{3}}}{5 \, 2^{\frac{2}{3}} \pi^{\frac{4}{3}}} . \qquad \cdots \quad (46)$$

On the other hand for the relativistic case ($s=$, we have, to a first order,

$$\left(\frac{\partial T}{\partial V} \right)_{E_-} = \frac{T}{V} \frac{(6A_0^R)^{\frac{2}{3}}}{4\pi^2} . \qquad \cdots \quad (47)$$

It is clear from equation (46) and (47) that for a Fermi-Dirac gas there is a heating effect in the degenerate case also, and that this effect is greater the greater the degree of degeneracy.

For degenerate matter obeying Bose-Einstein statistics the expressions for p and C_v are given by equations (38) and (39) and hence substituting them in equation (42), we have (the exact expression)

$$\left(\frac{\partial T}{\partial V} \right)_{E_-} = -\frac{T}{(s+1)V} . \qquad \cdots \quad (48)$$

Hence, in the non-relativistic case ($s = 3/2$),

$$\left(\frac{\partial T}{\partial V} \right)_{E_-} = -\frac{2}{5} \frac{T}{V},$$

and, in the relativistic case ($s = 3$),

$$\left(\frac{\partial T}{\partial V} \right)_{E_-} = -\frac{1}{4} \frac{T}{V} . \qquad \cdots \quad (50)$$

Equations (43) and (48) show that for matter obeying Bose Einstein statistics Joule effect always produces cooling.

As in the case of the Joule-Thomson effect the equation (43) for the non-degenerate case of Bose-Einstein statistics is only mildly convergent as $A_1 = \zeta(s) - 0$, and does not give sufficiently accurate values of $\left(\frac{T}{\partial V} \right)$

as the gas approaches the state of degeneracy. We proceed as before to find out the nature of the curve showing the variation of $\left(\frac{\partial T}{\partial V}\right)\Big/\frac{T}{V}$ with A_1 at $A_1=\zeta(s)-0$ by evaluating $\frac{\partial T}{\partial V}\Big/\frac{T}{V}$ and its differential coefficient with respect to A_1 at the point $A_1=\zeta(s)-0$. We have from equation (14)

$$p_{*}=\frac{R}{V}\frac{T}{A_1}\Sigma\frac{A^n}{n^{s+1}}, \qquad \dots (51)$$

and therefore

$$p_1=\frac{RT_0}{V}\frac{\zeta(s+1)}{\zeta(s)}. \qquad \dots (52)$$

Substituting the values from (25) and (52) in (42) and simplifying, we obtain

$$\left(\frac{\partial T}{\partial V}\right)_1=-\frac{T_0}{V(s+1)}\left\{\frac{1-\dfrac{[\zeta(s)]^2}{\zeta(s+1)\zeta(s-1)}}{1-\dfrac{s}{s+1}\dfrac{[\zeta(s)]^2}{\zeta(s+1)\zeta(s-1)}}\right\}, \qquad \dots (53)$$

which gives, for the non-relativistic case $(s=3/2)$,

$$\left(\frac{\partial T}{\partial V}\right)_1=-0{\cdot}4\frac{T_0}{V}, \qquad \dots (54)$$

and, for the relativistic case $(s=3)$,

$$\left(\frac{\partial T}{\partial V}\right)_1=-0{\cdot}12\frac{T_0}{V}. \qquad \dots (54a)$$

The slope of the curve at the limiting point is calculated in a similar way. We have

$$\frac{d}{dA_1}\left(\frac{\partial T}{\partial V}\right)_E=\frac{d}{dA_1}\left(\frac{p}{C_v}-\frac{T}{sV}\right).$$

Substituting the values of p and C_v from equations (51) and (24), we get

$$\frac{d}{dA_1}\left(\frac{\partial T}{\partial V}\Big/\frac{T}{V}\right)=\frac{1}{s}\frac{d}{dA_1}\left[\frac{\dfrac{1}{A_1}\Sigma\dfrac{A^n}{n^{s+1}}}{\left\{\dfrac{(s+1)}{A_1}\Sigma\dfrac{A^n}{n^{s+1}}-s\dfrac{\Sigma\dfrac{A^{n-1}}{n^s}}{\Sigma\dfrac{A^{n-1}}{n^{s-1}}}\right\}}-1\right]$$

$$=\frac{1}{s}\frac{d}{dA_1}\left[\frac{1}{(s+1)-sA_1^2\Big/\Sigma\dfrac{A^n}{n^{s+1}}\Sigma\dfrac{A^n}{n^{s-1}}}-1\right] \qquad \dots (55)$$

which on differentiation and simplification using equation (20*a*) gives

$$\left[\frac{d}{dA_1}\left(\frac{\partial T}{\partial V}\Big/\frac{T}{V}\right)\right]_1 = -\frac{\dfrac{[\zeta(s)]^2\zeta(s+1)\zeta(s-2)}{[\zeta(s-1)]^3} - \dfrac{2\zeta(s)\zeta(s+1)}{\zeta(s-1)} + \dfrac{[\zeta(s)]^3}{[\zeta(s-1)]^2}}{\left\{(s+1)\zeta(s+1) - \dfrac{s[\zeta(s)]^2}{\zeta(s-1)}\right\}^2} \quad \ldots\ldots (56)$$

In the non-relativistic case, *i.e.*, $s=3/2$, this gives us

$$\left[\frac{d}{dA_1}\left(\frac{\partial T}{\partial V}\Big/\frac{T}{V}\right)\right]_1 = -0{\cdot}1295, \qquad \ldots (57)$$

and in the relativistic case, *i.e.*, $s=3$,

$$\left[\frac{d}{dA_1}\left(\frac{\partial T}{\partial V}\Big/\frac{T}{V}\right)\right]_1 = - \quad . \qquad \ldots (58)$$

The curves showing the variation of $\left(\dfrac{\partial T}{\partial V}\Big/\dfrac{T}{V}\right)$ with A_1 for a Bose-Einstein gas have been drawn in figure 2. The ordinate represents $\left(-\dfrac{\partial T}{\partial V}\Big/\dfrac{T}{V}\right)$ and the abscissa represents A_1. The curve NN is for the

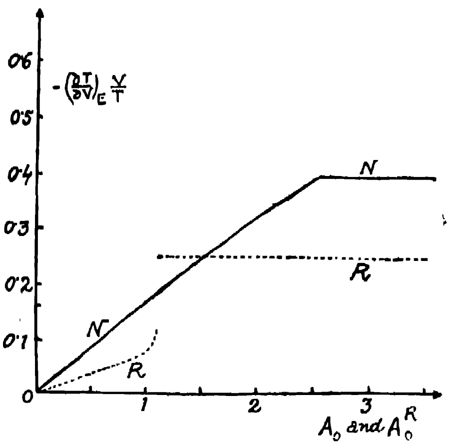

FIGURE 2

Joule effect for a Bose-Einstein gas. Curve NN is for the non-relativistic case and RR for the relativistic case.

non-relativistic case ($A_1=A_0$) and RR for the relativistic case ($A_1=A_0^r$). The discontinuity in the slope of the curves at $A_1=\zeta(s)$ has been determined from equation (57) and (58) which give the angles to be $7°\ 23'$ nearly in the non-relativistic case, and $90°$ in the relativistic case. As the ratio of the scales of the axes in the figure is $5:1$, the angles correspond to about

$33°$ for curve NN and $90°$ for curve RR. It will be noticed, as is readily shown from equations (48) and (53), that there is no discontinuity in the magnitude of the Joule effect in a Bose-Einstein gas as it passes from non-degeneracy to degeneracy in the non-relativistic case, *i.e.*, at $A_0 = \zeta\,(3/2)$. On the other hand there is a discontinuity in the relativistic case at $A_0^n = \zeta(3)$.

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